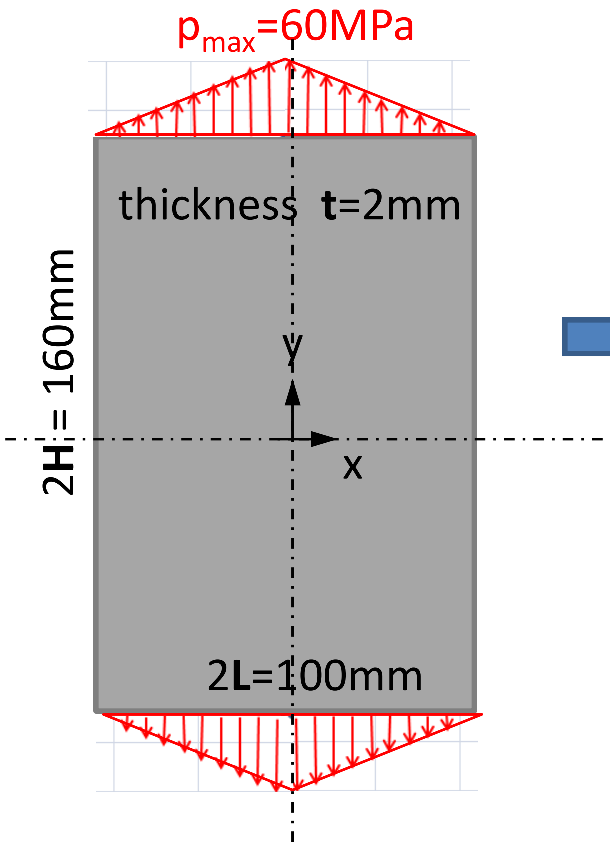
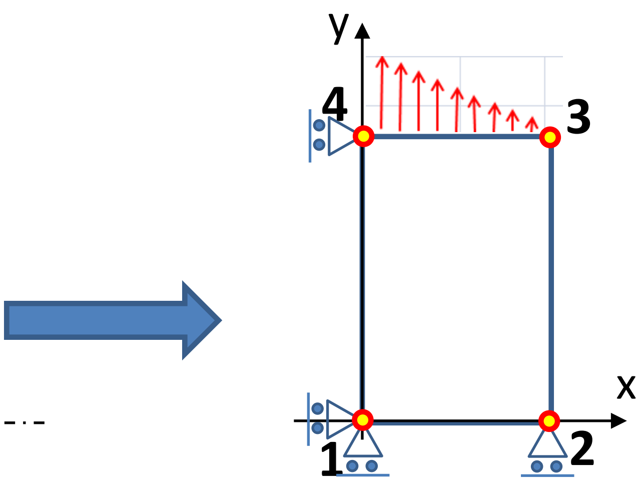
Finite element method (FEM1)

Lecture 6C. Plate 2D modeled with 4-node elements

03.2025

Example. 2D plate modeled using 4-node element





one-quarter model with single 4-node element

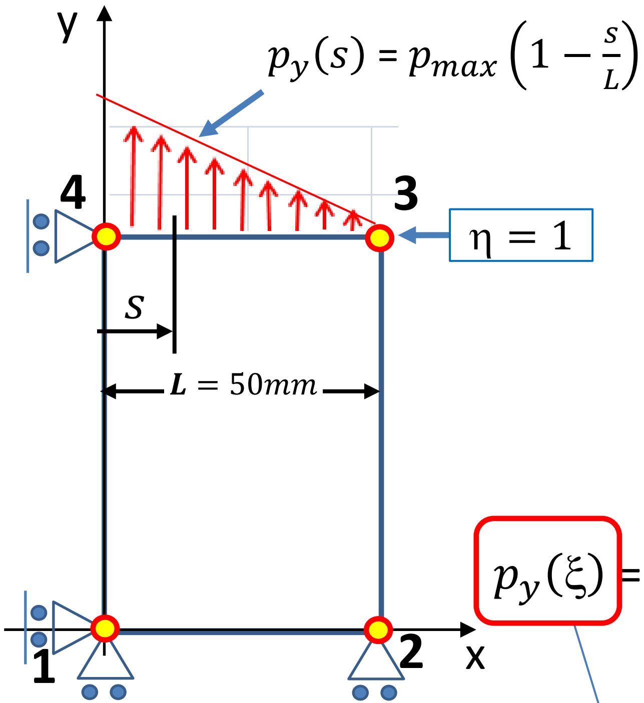
vectors of nodal coordinates:

$$\lfloor x_i \rfloor_1 = \lfloor 0, L, L, 0 \rfloor$$

$$\lfloor y_i \rfloor_1 = \lfloor 0, 0, H, H \rfloor$$

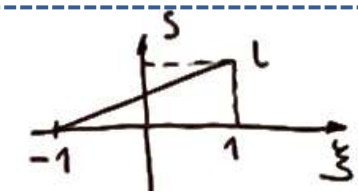
Equivalent load vector due to surface forces

$$\underset{1\times 8}{\lfloor F^p \rfloor_e} = t_e \int_{-1}^{1} \lfloor p_x, p_y \rfloor \underset{2\times 8}{[N]} \sqrt{\left(\underset{1\times 4}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \; \underset{4\times 1}{\{x_i\}_e}\right)^2 + \left(\underset{1\times 4}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \; \underset{4\times 1}{\{y_i\}_e}\right)^2} \, d\xi$$



$$x_3 = L \,,\; x_4 = 0$$
$$y_3 = H \,,\; y_4 = H$$

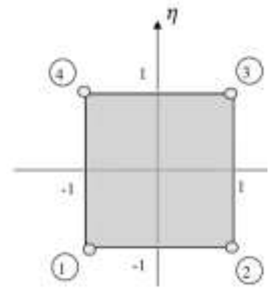
$$\left.\begin{array}{l} s=0 \;\rightarrow\; \xi = -1 \\ s=L \;\rightarrow\; \xi = +1 \end{array}\right\} \rightarrow s(\xi) = \frac{L}{2}(\xi + 1)$$



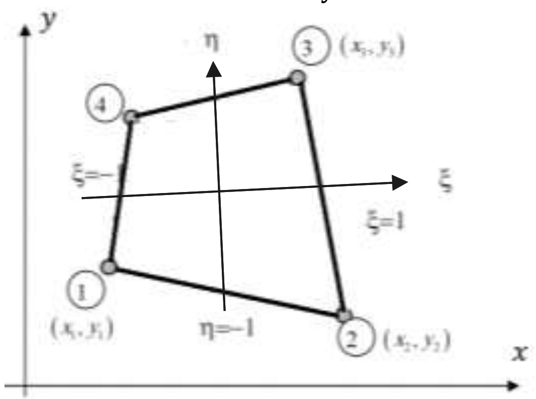
$$\boxed{p_y(\xi)} = p_{max}\left(1 - \frac{s(\xi)}{L}\right) = p_{max}\left(1 - \frac{1}{2}(\xi + 1)\right) = \boxed{\frac{p_{max}}{2}(1 - \xi)}$$

$$\underset{1\times 8}{\lfloor F^p \rfloor_1} = t \int_{-1}^{1} \lfloor 0, p_y(\xi) \rfloor \underset{2\times 8}{[N]} \sqrt{\left(\underset{1\times 4}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \; \underset{4\times 1}{\{x_i\}_e}\right)^2 + \left(\underset{1\times 4}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \; \underset{4\times 1}{\{y_i\}_e}\right)^2} \, d\xi$$

natural coordinate system



cartesien coordinate system



shape functions of the element and their derivatives in the natural system:

$$N_1(\xi,\eta)= ¼(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta)= ¼(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta)= ¼(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta)= ¼(1-\xi)(1+\eta)$$

i	$N_i(\xi,1)$
1	0
2	0
3	$½(1+\xi)$
4	$-½(1-\xi)$

i	$\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$-¼(1-\eta)$	$-¼(1-\xi)$
2	$¼(1-\eta)$	$-¼(1+\xi)$
3	$¼(1+\eta)$	$¼(1+\xi)$
4	$-¼(1+\eta)$	$¼(1-\xi)$

$\frac{\partial N_i(\xi,1)}{\partial \xi}$
$-¼(1-1)=$ **0**
$¼(1-1) =$ **0**
$¼(1+1)=$ **½**
$-¼(1+1)=$ **-½**

$$x_3 = L\,,\ x_4 = 0$$
$$y_3 = H\,,\ y_4 = H$$

$$\frac{\partial N_3(\xi,1)}{\partial \xi} = ½\,,\quad \frac{\partial N_4(\xi,1)}{\partial \xi} = -½$$

$$N_3(\xi,1) = ½(1+\xi)\quad,\quad N_4(\xi,1) = -½(1-\xi)$$

$$\lfloor F^p \rfloor_1 = t\int_{-1}^{1} \lfloor 0, p_y(\xi) \rfloor [N] \sqrt{\left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{x_i\}_e\right)^2 + \left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{y_i\}_e\right)^2}\, d\xi$$

$$\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{x_i\}_1 = \frac{\partial N_3(\xi,1)}{\partial \xi} x_3 + \frac{\partial N_4(\xi,1)}{\partial \xi} x_4 = ½ \cdot L - ½ \cdot 0 = \frac{L}{2}$$

$$\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{y_i\}_1 = \frac{\partial N_3(\xi,1)}{\partial \xi} y_3 + \frac{\partial N_4(\xi,1)}{\partial \xi} y_4 = ½ \cdot H - ½ \cdot \mathrm{H} = 0$$

$$\sqrt{\left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{x_i\}_e\right)^2 + \left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}\{y_i\}_e\right)^2} = \frac{L}{2}$$

$$\lfloor F^p \rfloor_1 = t\,\frac{L}{2}\int_{-1}^{1} \lfloor 0, p_y(\xi) \rfloor [N(\xi,1)]\, d\xi$$

$$[N(\xi,1)] = \begin{bmatrix} 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) & 0 \\ 0 & 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) \end{bmatrix}$$

$N_3(\xi,1)$= ½$(1+\xi)$, $N_4(\xi,1)$= - ½$(1-\xi)$

$$[N(\xi,1)] = \begin{bmatrix} 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) & 0 \\ 0 & 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) \end{bmatrix}$$

$$\lfloor F^p \rfloor_1 = t\frac{L}{2}\int_{-1}^{1} \lfloor 0, p_y(\xi) \rfloor [N(\xi,1)]\, d\xi$$

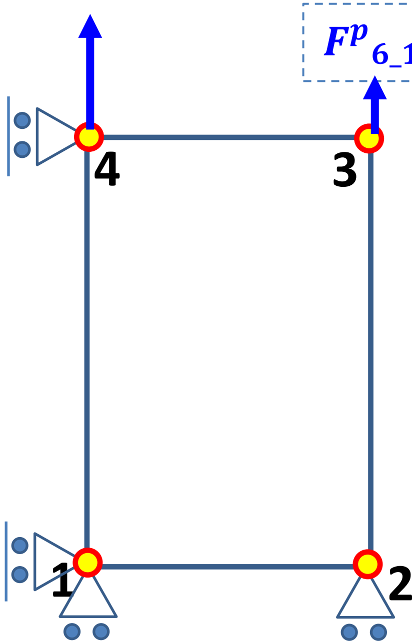
$$p_y(\xi) = \frac{p_{max}}{2}(1-\xi)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) & 0 \\ 0 & 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) \end{bmatrix}$$

$$\lfloor 0\,, p_y(\xi) \rfloor \quad \lfloor 0,0,0,0,0, \underbrace{N_3(\xi,1)p_y(\xi)}_{F^p{}_{6\_1}}, 0, \underbrace{N_4(\xi,1)p_y(\xi)}_{F^p{}_{8\_1}} \rfloor$$

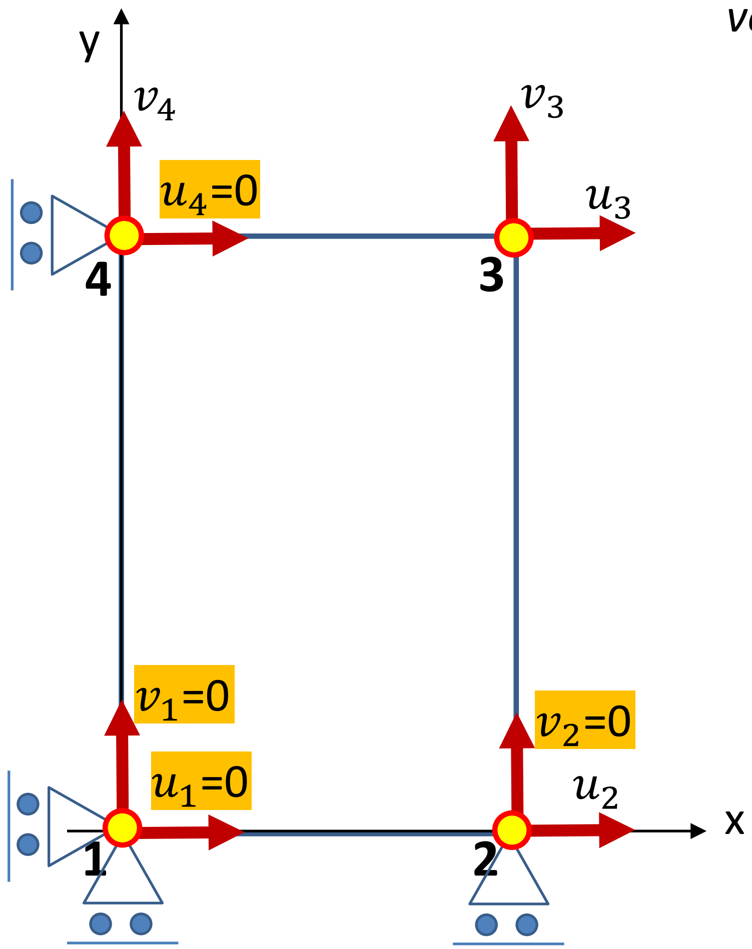
$$F^p{}_{8\_1} = \frac{tL}{3} p_{max}$$

$$F^p{}_{6\_1} = \frac{tL}{6} p_{max}$$



$$F^p{}_{6\_1} = t\frac{L}{2}\int_{-1}^{1} \frac{p_{max}}{2}(1-\xi)\,½(1+\xi)\, d\xi = \frac{tL}{6} p_{max}$$

$$F^p{}_{8\_1} = t\frac{L}{2}\int_{-1}^{1} \frac{p_{max}}{2}(1-\xi)\big(-½(1-\xi)\big)\, d\xi = \frac{tL}{3} p_{max}$$



vector of element nodal parameters

$$\underset{8\times 1}{\{q\}_1} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_1$$

column vector

$$\underset{1\times 8}{\lfloor q \rfloor_1} = \lfloor u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 \rfloor_1$$

row wector

Boundary conditions: $\mathrm{N} = 8 - 4 = 4$

vector of active nodal parameters after taking into account boundary conditions :

$$\underset{1\times 4}{\lfloor q \rfloor_1} = \lfloor u_2, u_3, v_3, v_4 \ \rfloor_1$$

Gradient matrix for the plain stress condition:

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\right) & 0 \\ 0 & \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) \\ \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\right) \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right) & \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\right) & 0 \\ 0 & \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\right) \\ \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\right) \end{bmatrix}\frac{\partial}{\partial \xi} + \begin{bmatrix} \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\right) & \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\right) \end{bmatrix}\frac{\partial}{\partial \eta}$$

strain–displacement matrix

$$\boxed{\underset{3\times 16}{[B]} = \underset{3\times 2}{[R]}\underset{2\times 16}{[N]} = \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\right) & 0 \\ 0 & \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\right) \\ \left(-\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\right) \end{bmatrix}\left[\frac{\partial N}{\partial \xi}\right] + \begin{bmatrix} \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\right) & \left(-\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\right) \end{bmatrix}\left[\frac{\partial N}{\partial \eta}\right]}$$

$$det[J] = det\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} =$$

$$= \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \Bigg(\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{4\times 1}{\{x_i\}_e}\Bigg)\Bigg(\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{4\times 1}{\{y_i\}_e}\Bigg) - \Bigg(\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{4\times 1}{\{y_i\}_e}\Bigg)\Bigg(\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{4\times 1}{\{x_i\}_e}\Bigg)$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{4\times 1}{\{y_i\}_e}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{4\times 1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{4\times 1}{\{x_i\}_e}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times 4}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{4\times 1}{\{x_i\}_e}$$

$$\underset{2\times 8}{\left[\frac{\partial N}{\partial \xi}\right]} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \end{bmatrix}$$

$$\underset{2\times 8}{\left[\frac{\partial N}{\partial \eta}\right]} = \begin{bmatrix} \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$\underset{3\times 3}{[D]} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$constitutive\ matrix\ for\ plain\ stress$

element stiffness matrix:

$$\underset{8\times 8}{[k]_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B(\xi,\eta)]} det[J]\, d\xi d\eta$$

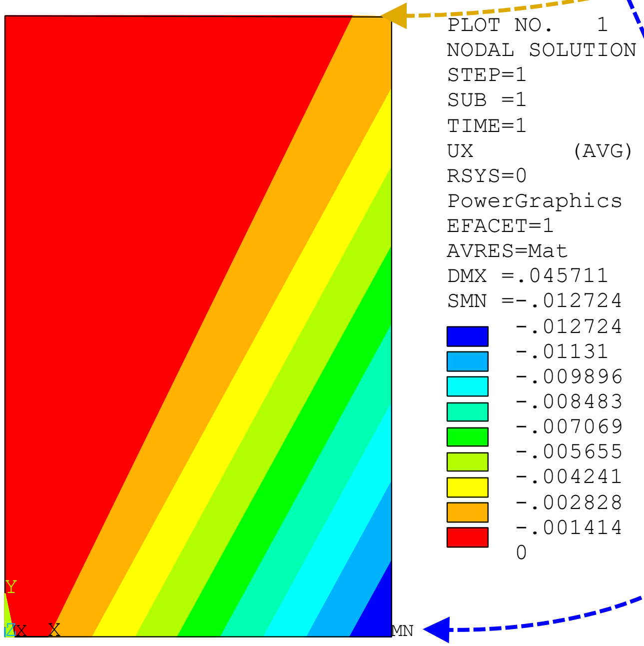
$numerical\ integration\ by\ the\ mixed\ quadrature\ rule$

to avoid „shear locking” (Enhanced strain formulation)

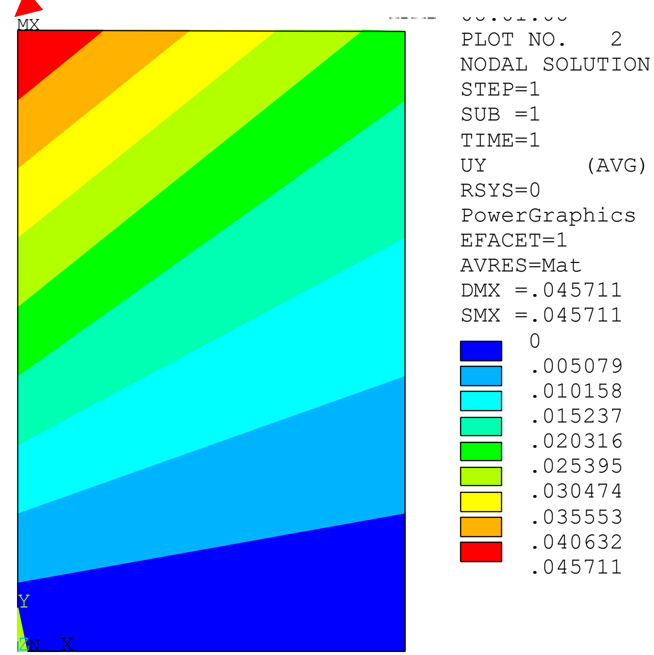
$$\underset{4\times1}{\{q\}} = \underset{4\times4}{[k]^{-1}} \; \underset{4\times1}{\{F\}}$$

F		q	
0		-.012724	u2
0		-.001562	u3
1000	N	.02286	v3
2000	N	.045711	v4

UX displacement

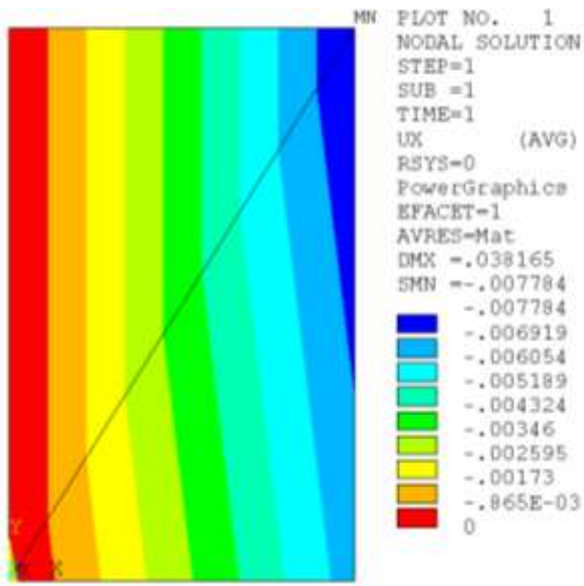


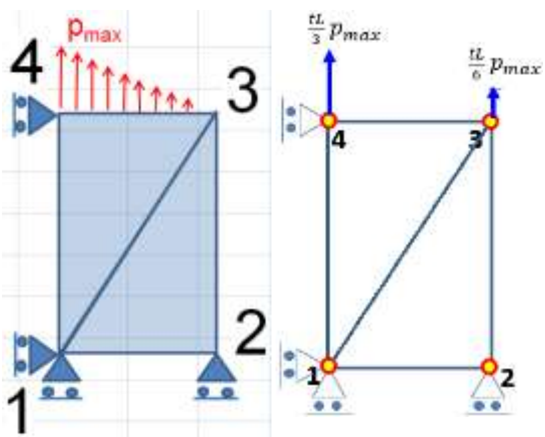
UY displacement

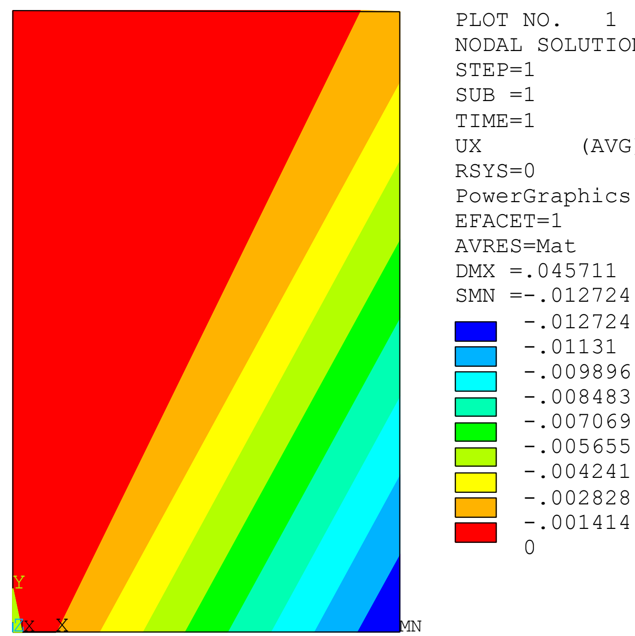


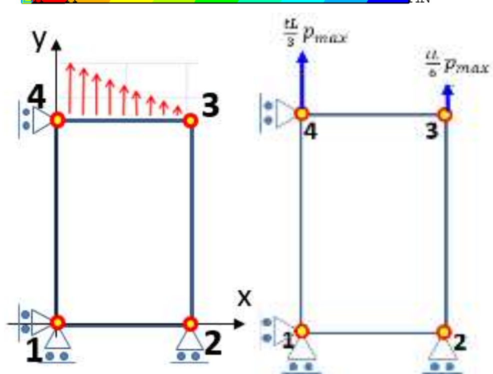
Displacements in X direction

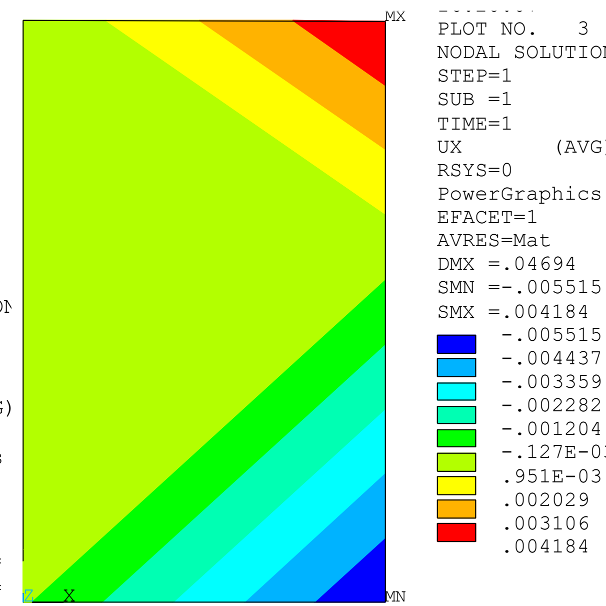
UX displacement

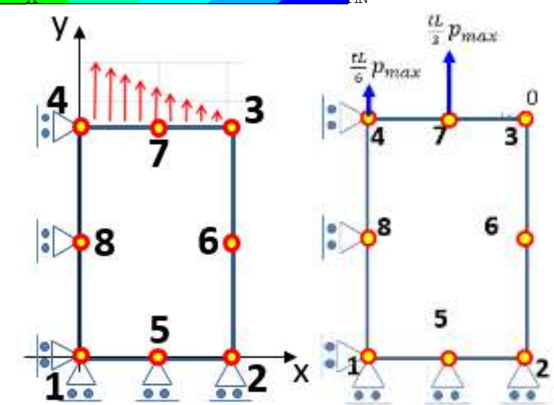






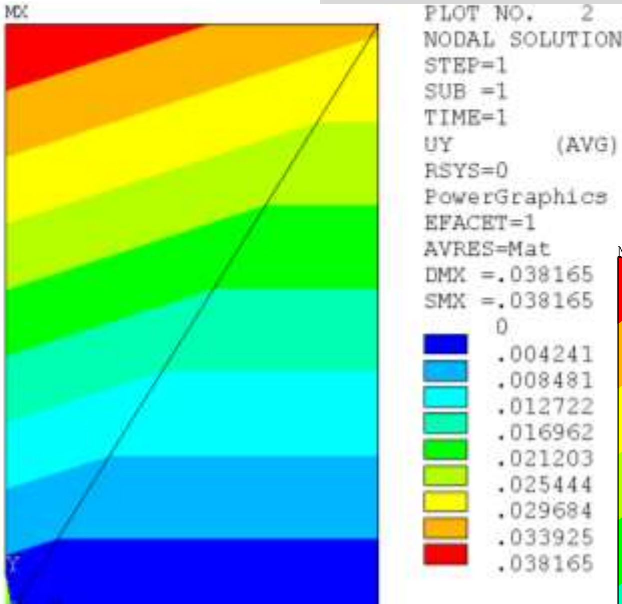


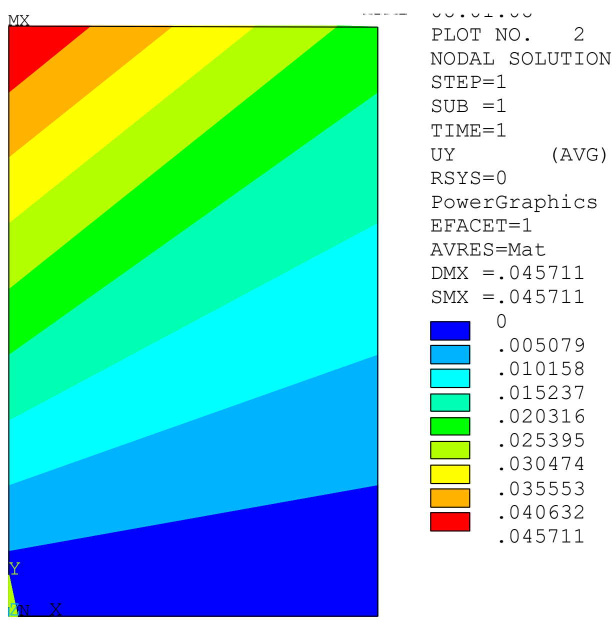


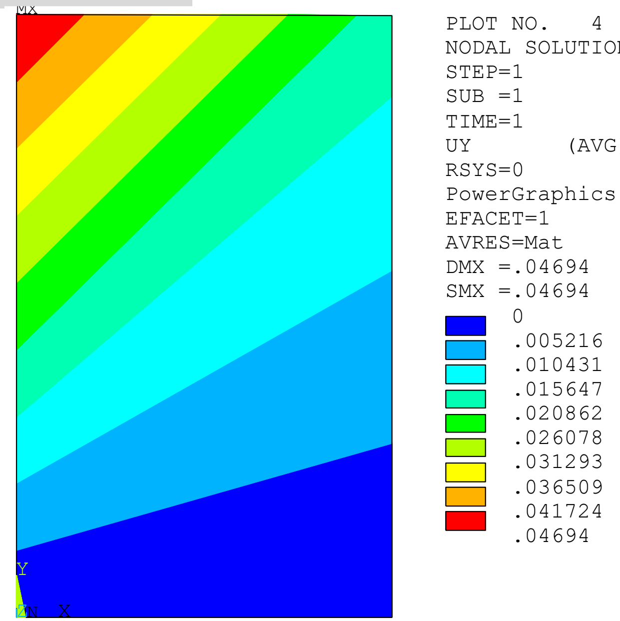


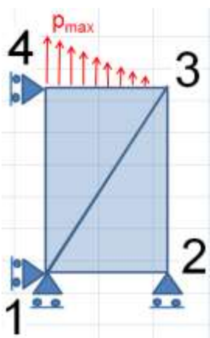
Displacements in Y direction

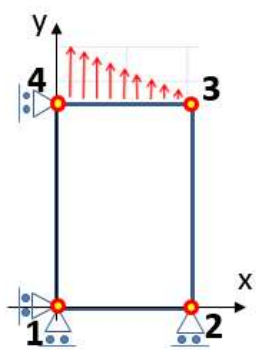
UY displacement

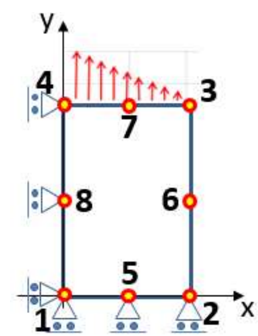






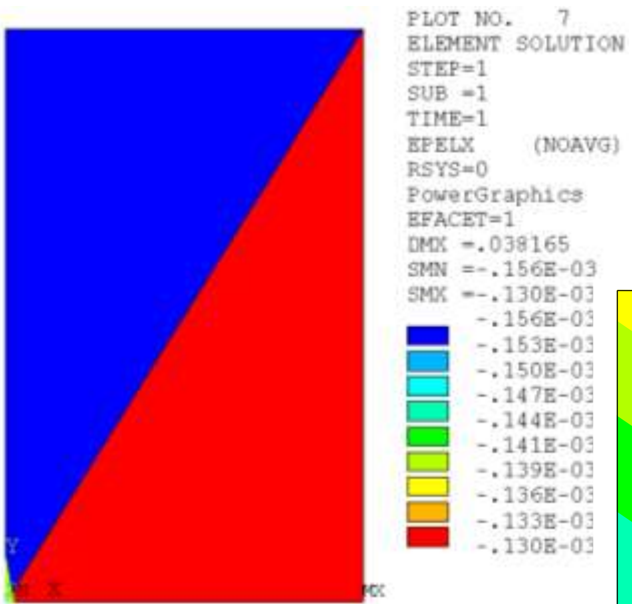


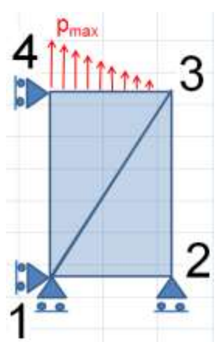


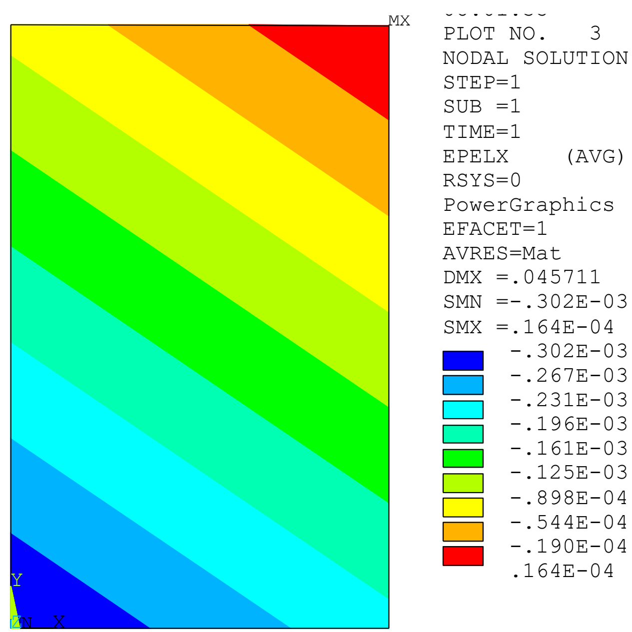


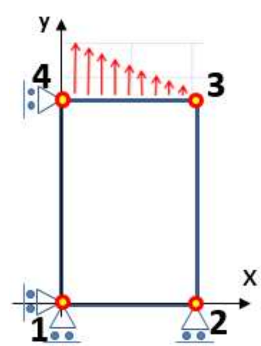
Strain in X direction

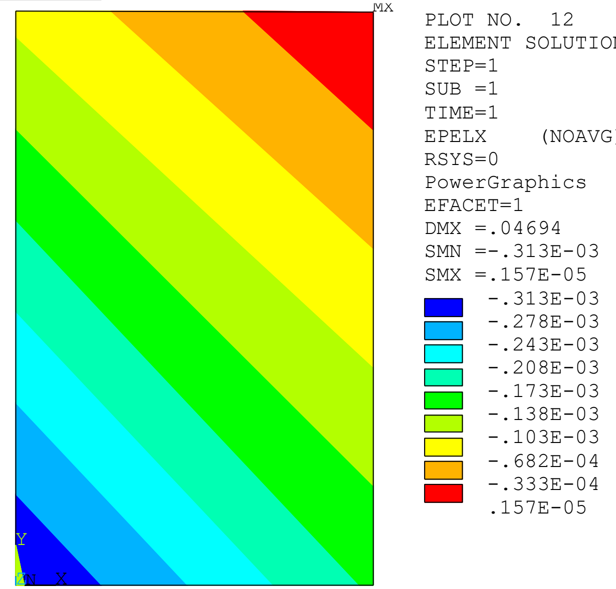
ε_x **strain**

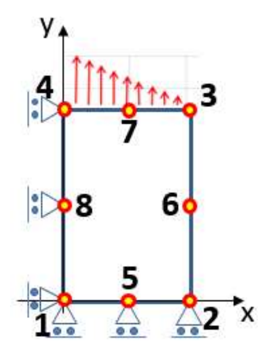




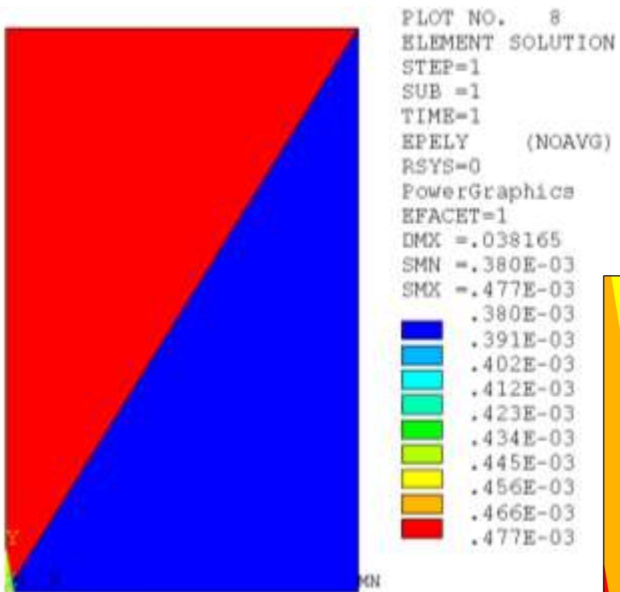


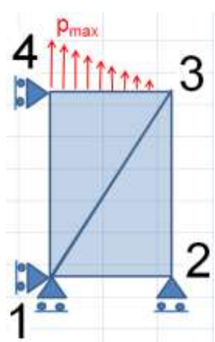




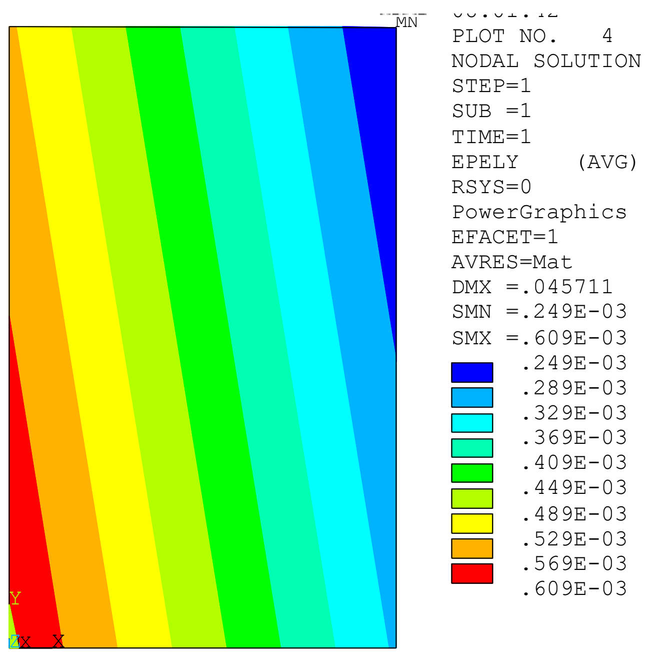


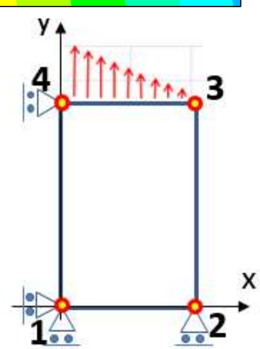
Strain in Y direction

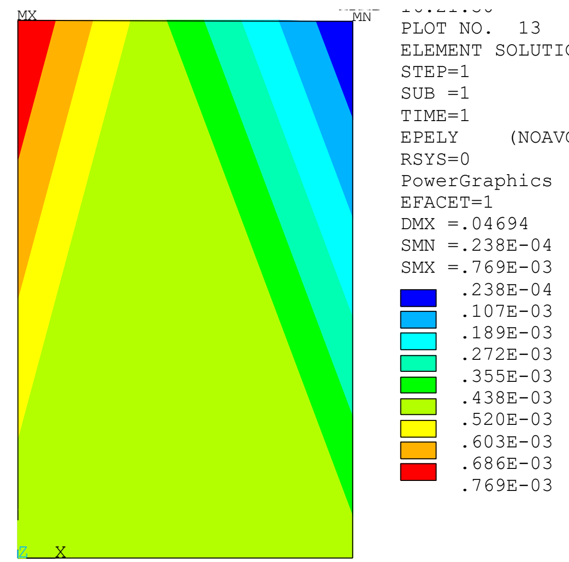


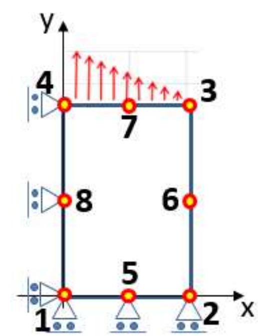


ε_y strain



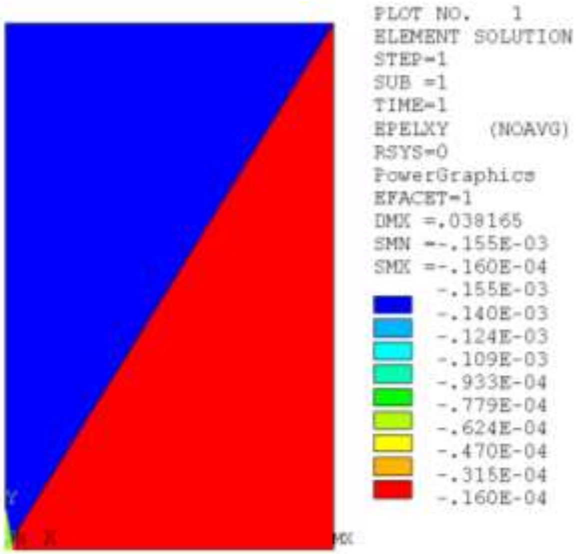


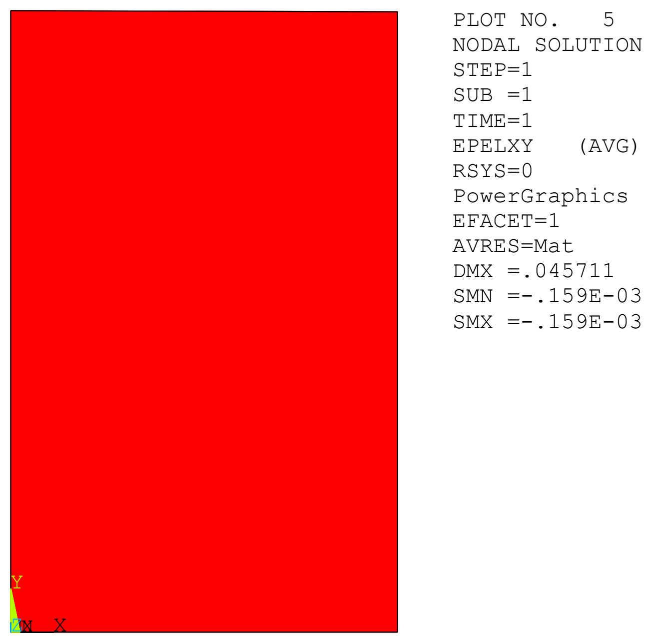


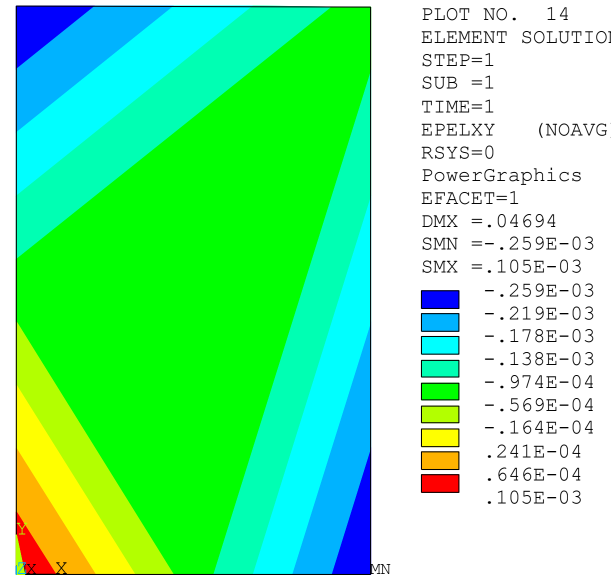


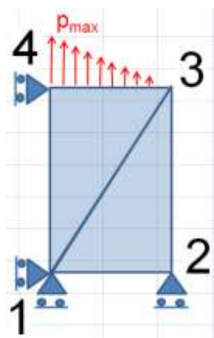
Shear strain

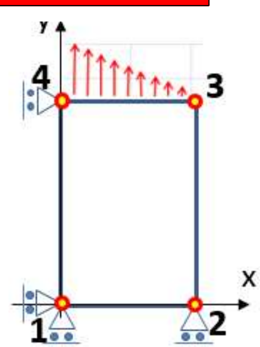
γ_{xy} **strain**

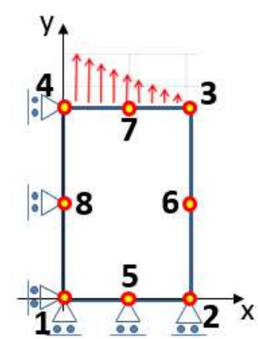






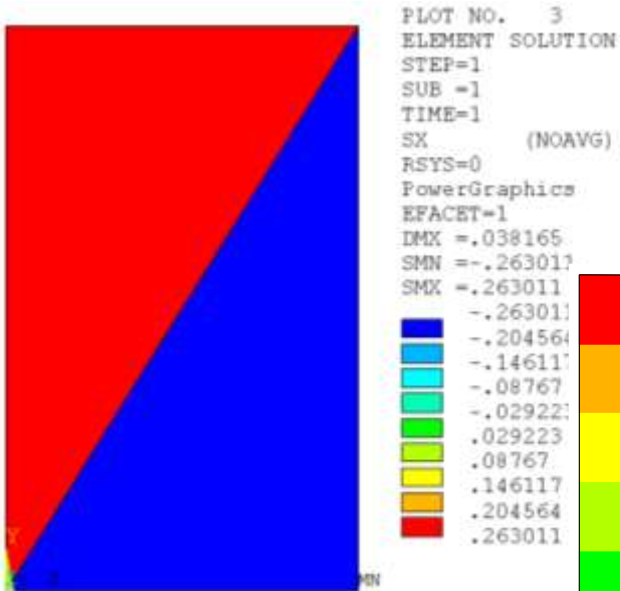


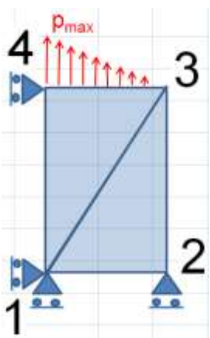


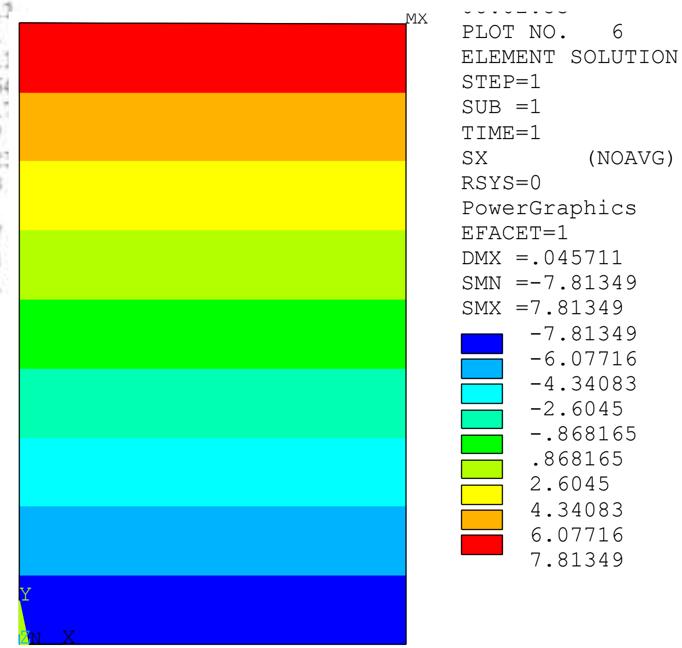


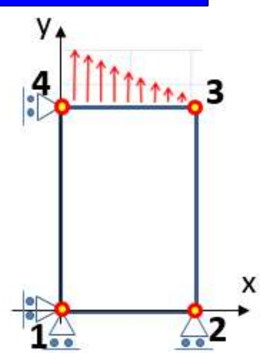
Stress in X direction

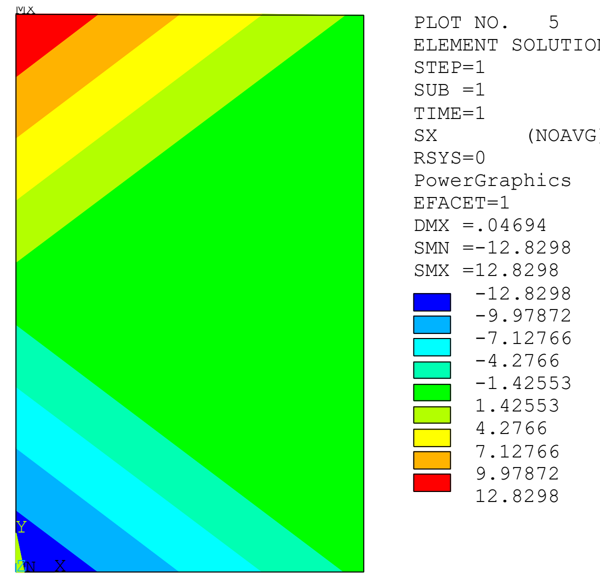
σ_x stress

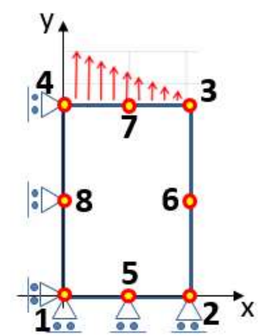




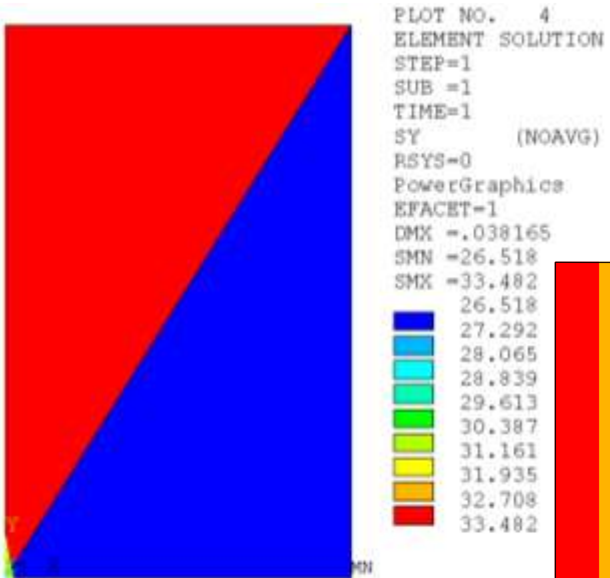




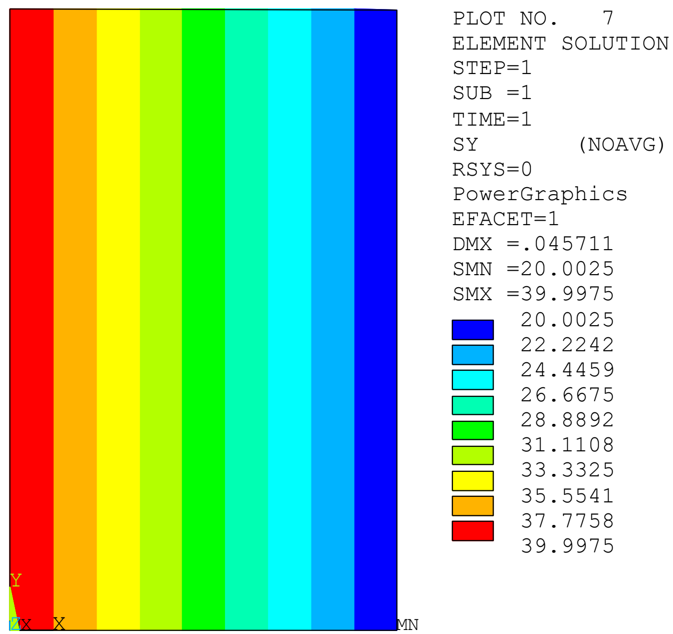


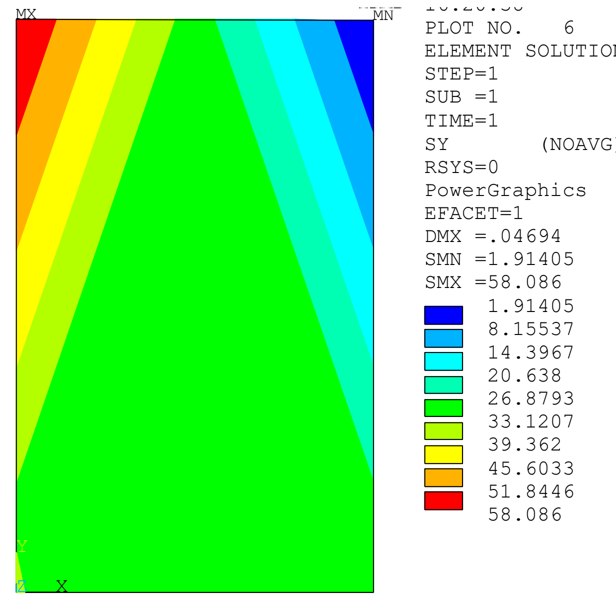


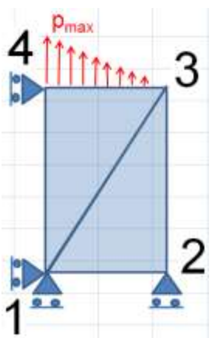
Stress in Y direction

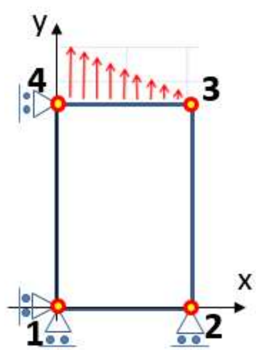


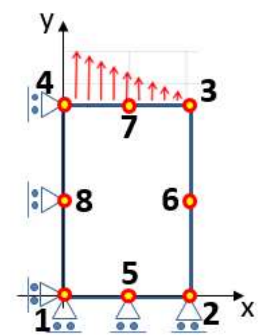
σ_y stress





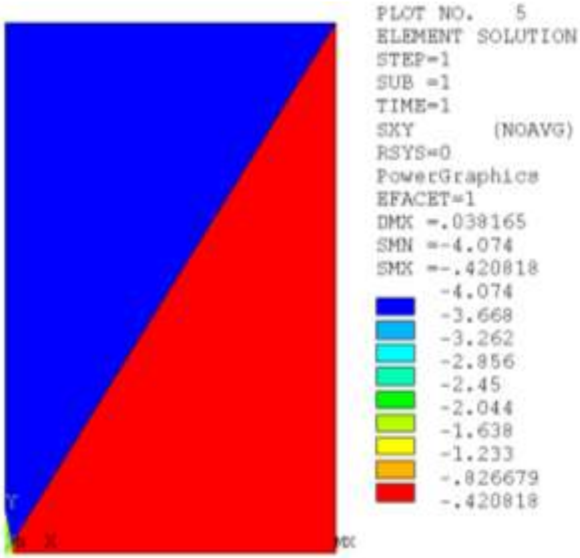


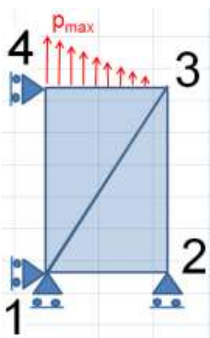


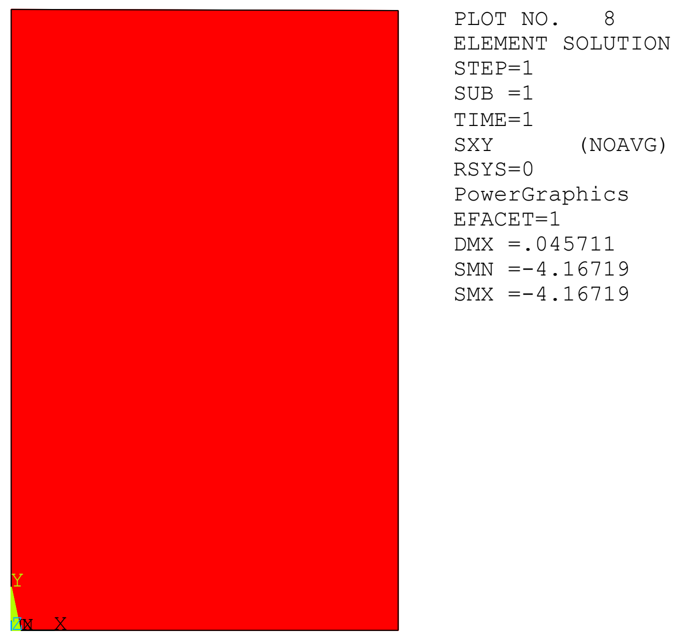


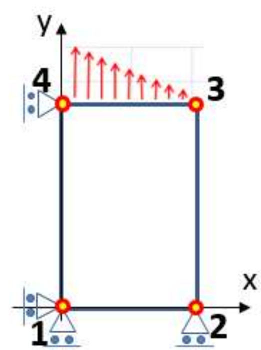
Shear stress

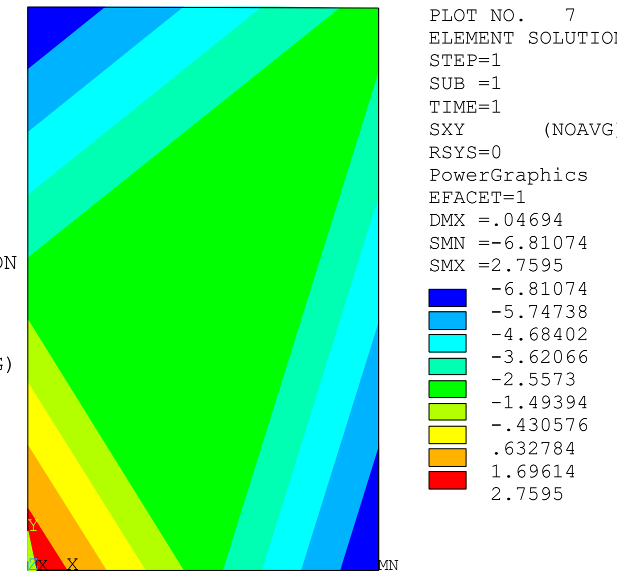
τ_{xy} stress

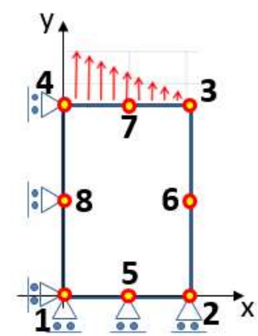




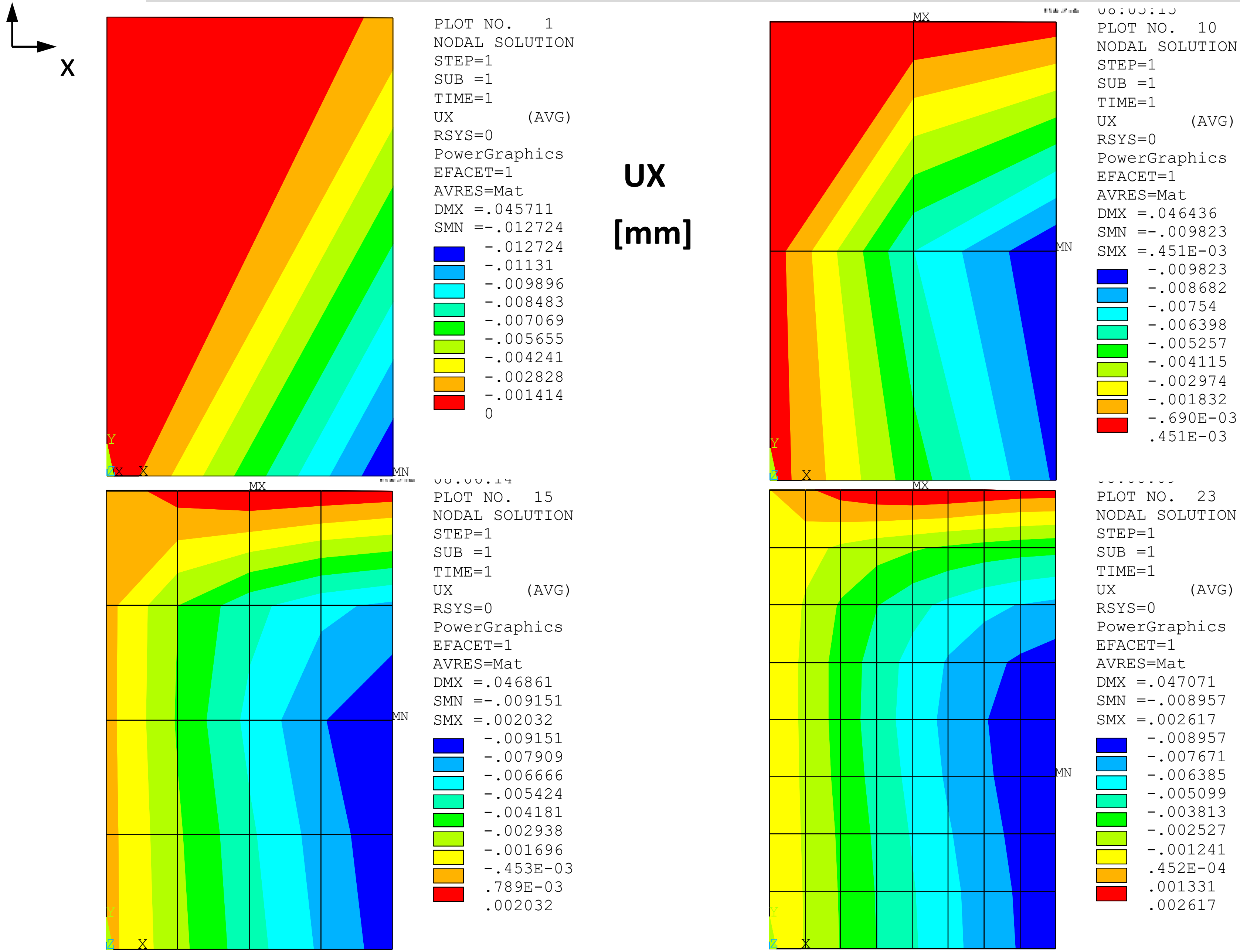




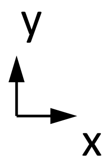


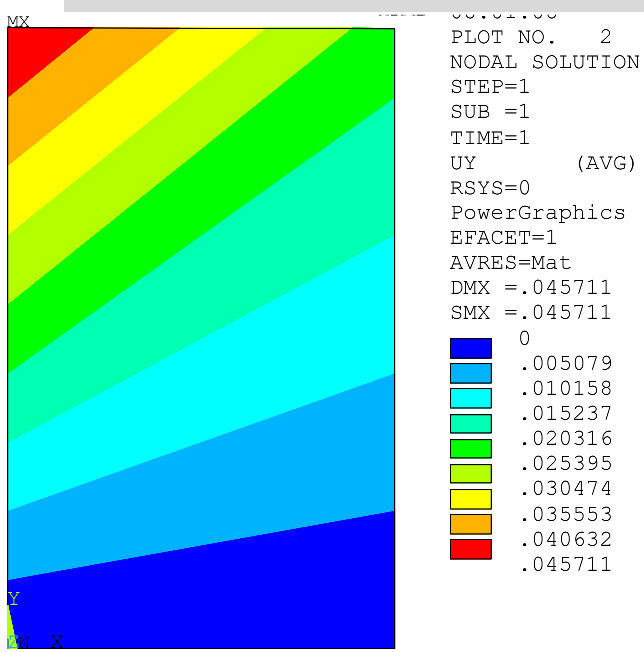


The impact of discretization on the quality of results *(4–node elements)*

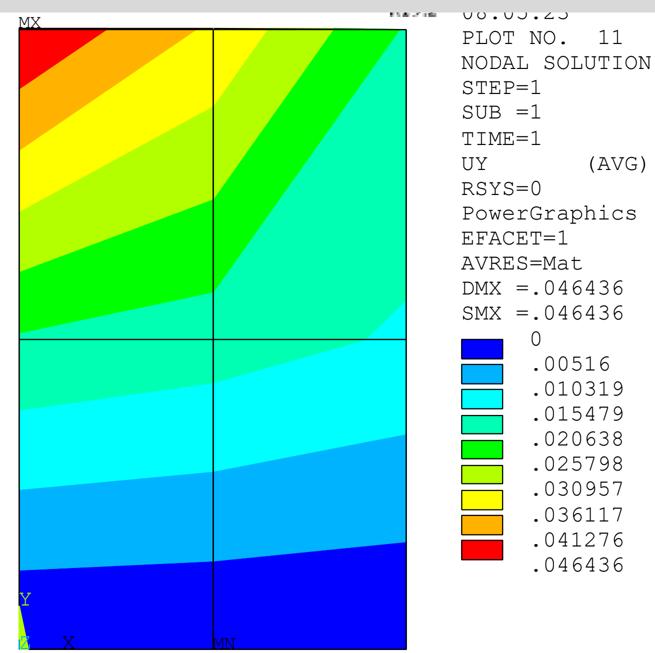


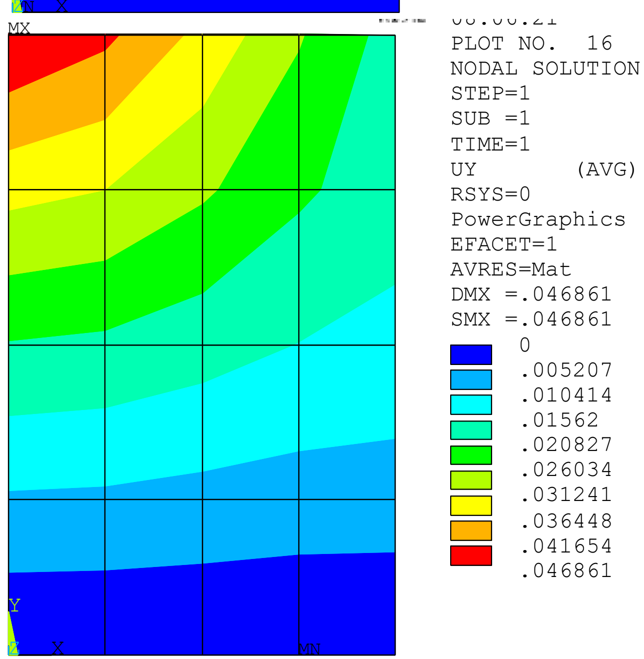
The impact of discretization on the quality of results *(4–node elements)*

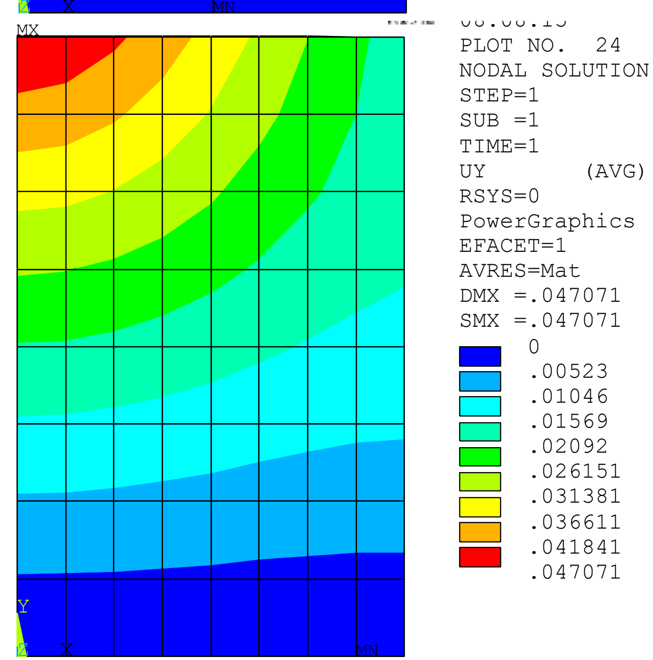




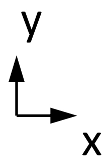
UY [mm]

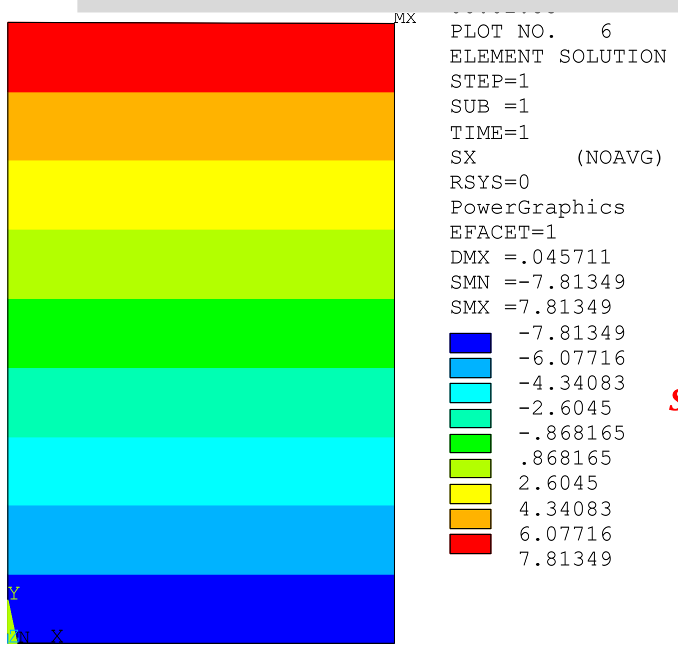






The impact of discretization on the quality of results *(4–node elements)*

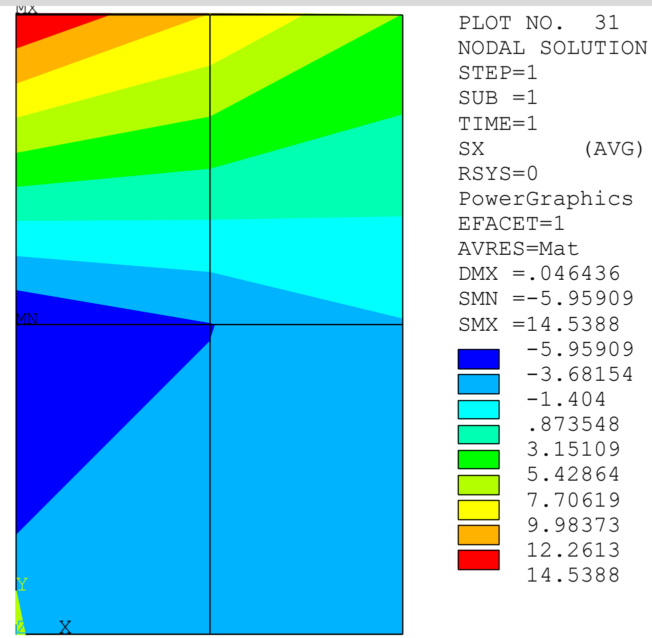


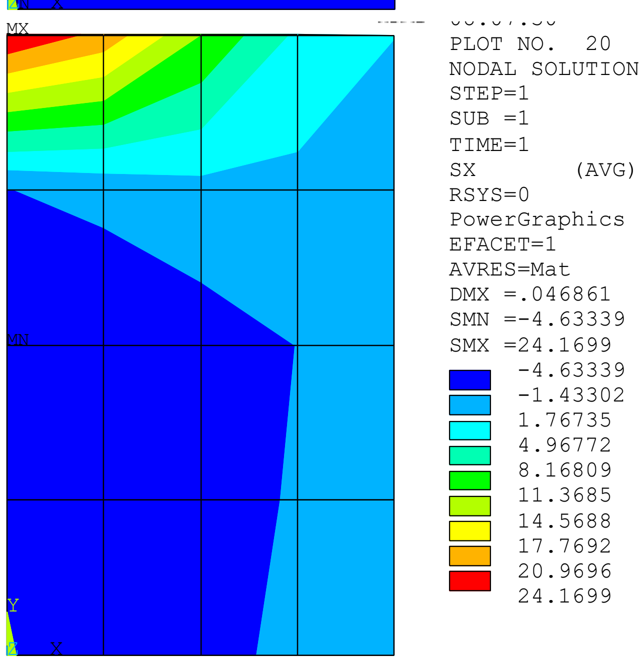


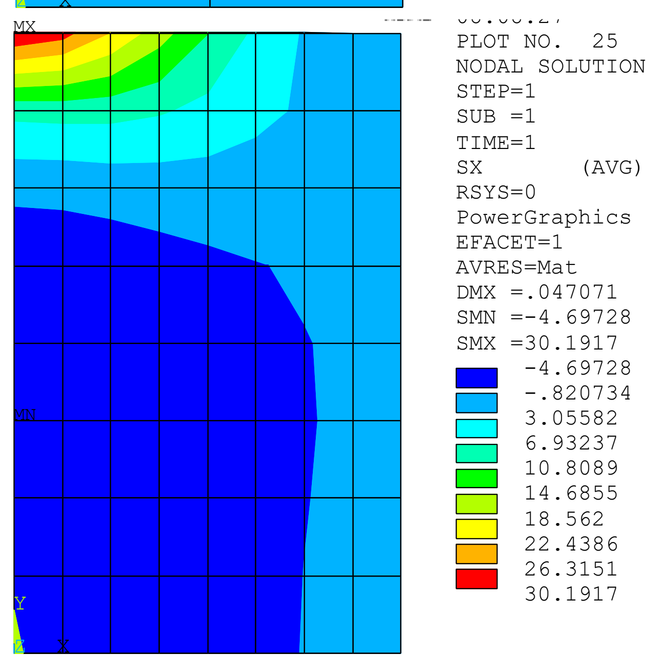
σ_x

[MPa]

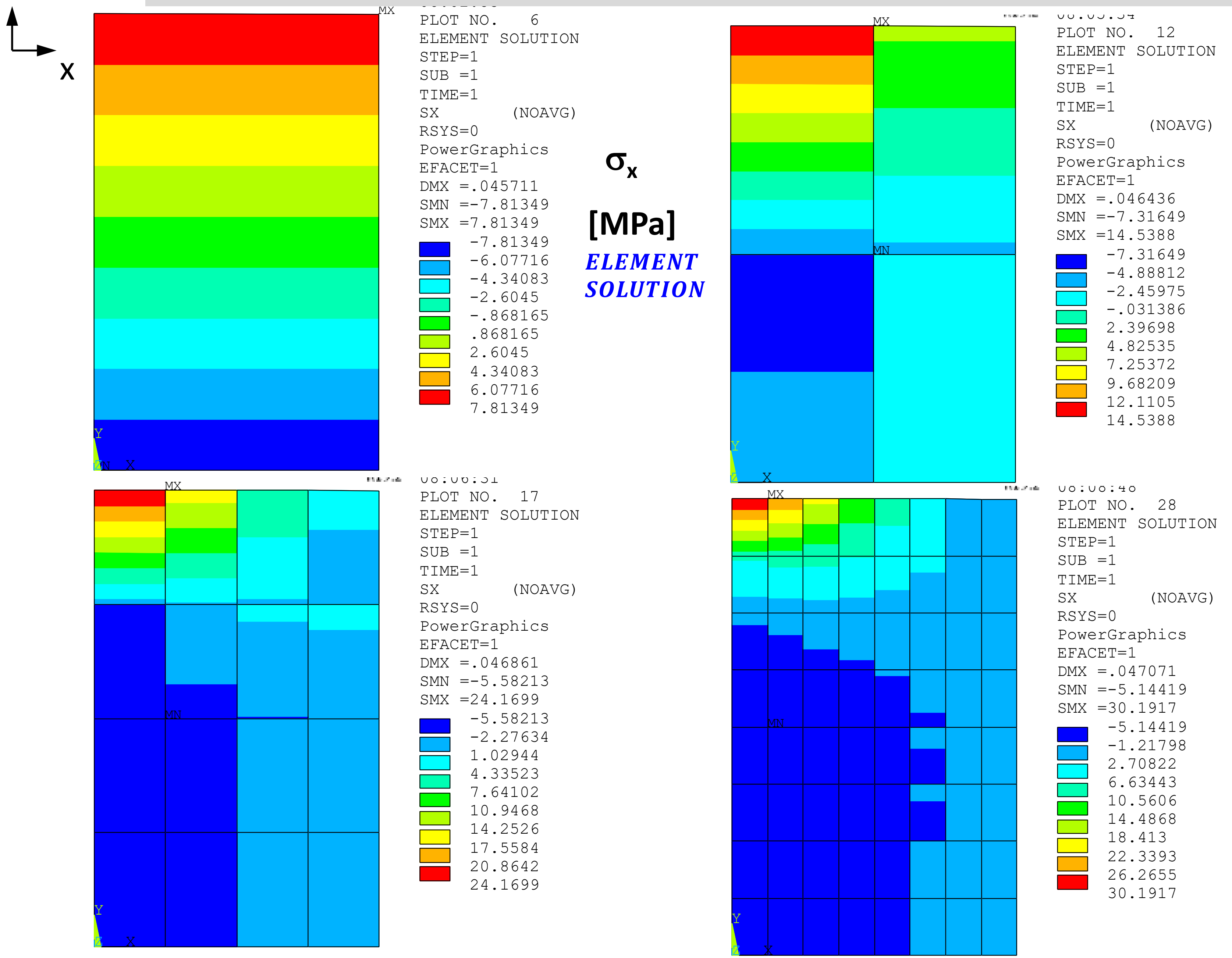
NODAL SOLUTION



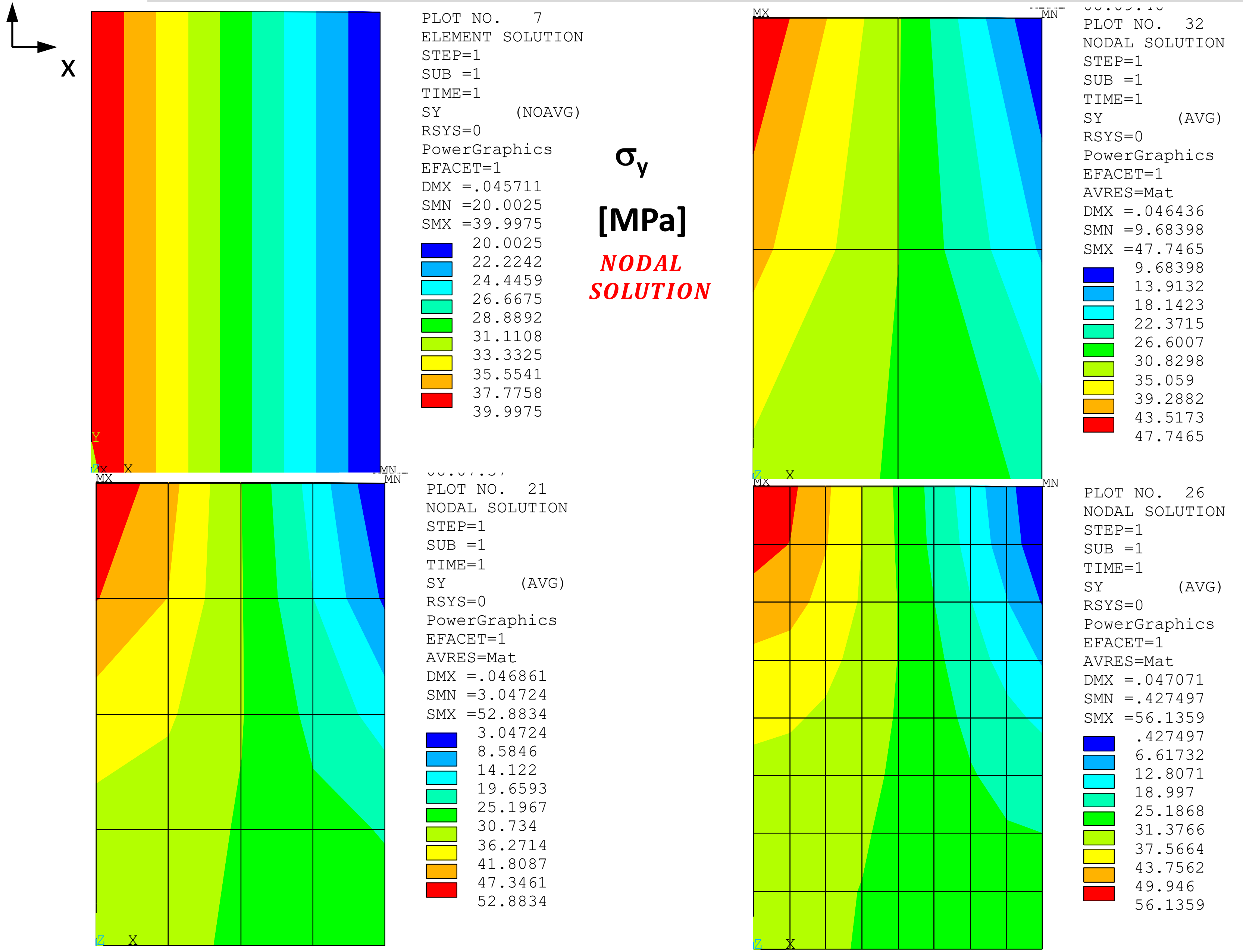




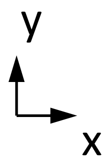
The impact of discretization on the quality of results *(4–node elements)*

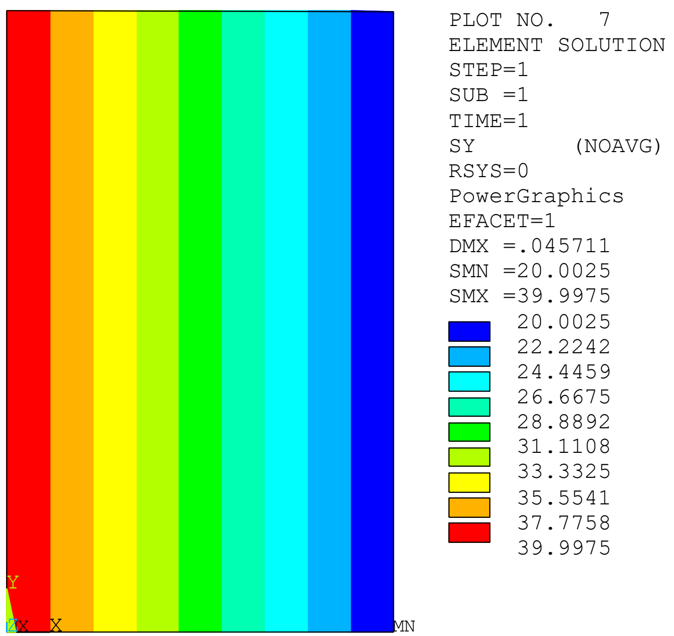


The impact of discretization on the quality of results *(4–node elements)*



The impact of discretization on the quality of results *(4–node elements)*

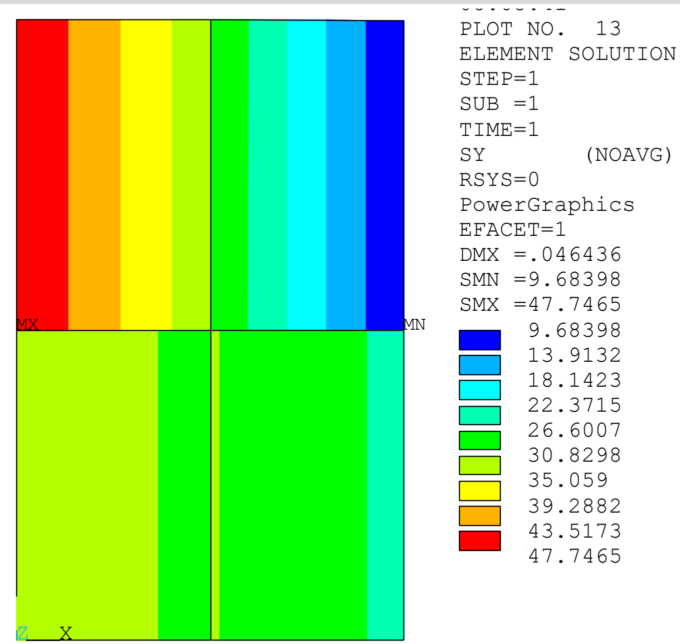


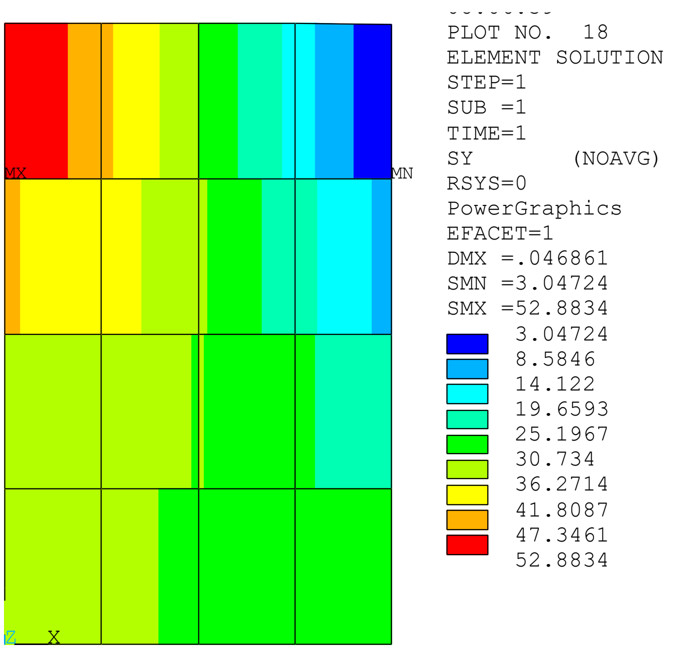


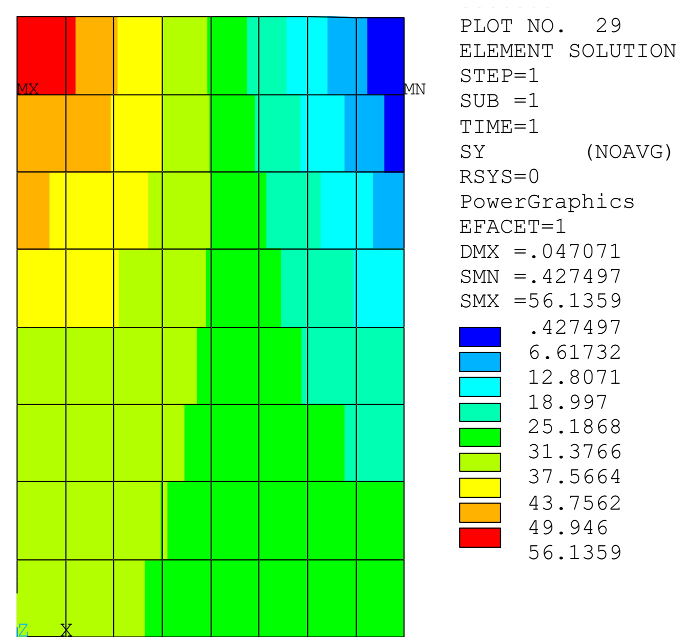
σ_y

[MPa]

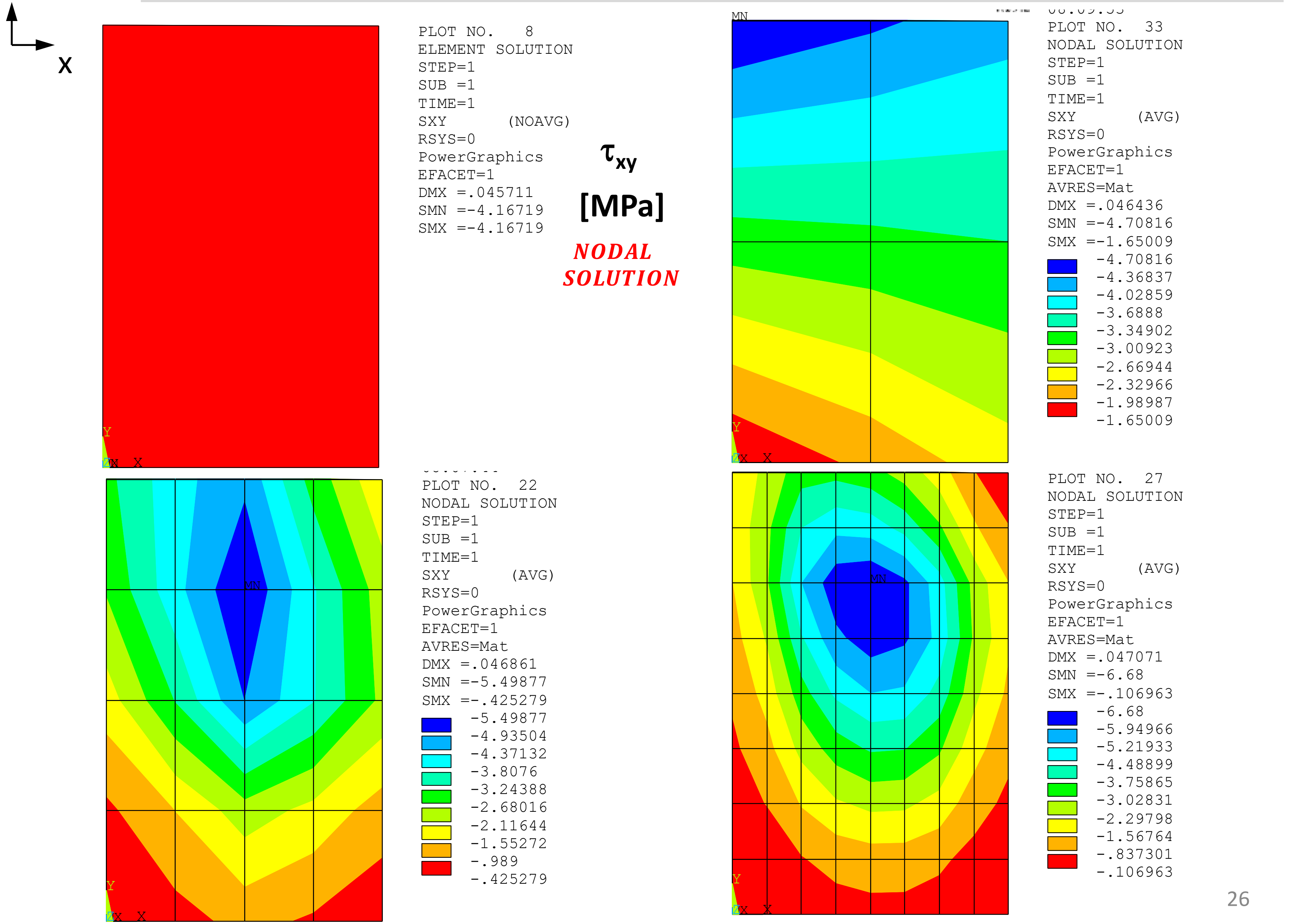
ELEMENT SOLUTION



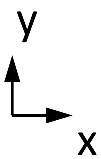


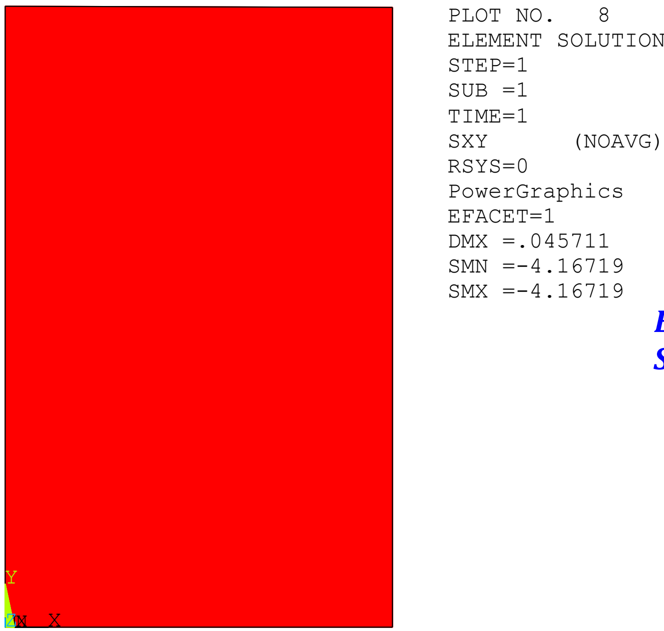


The impact of discretization on the quality of results *(4–node elements)*



The impact of discretization on the quality of results *(4–node elements)*





τ_{xy}

[MPa]

ELEMENT SOLUTION